

# The Alexander method for infinite-type surfaces

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## Abstract

We prove that for any infinite-type orientable surface  $S$  there exists a collection of essential curves  $\Gamma$  in  $S$  such that any homeomorphism that preserves the isotopy classes of the elements of  $\Gamma$  is isotopic to the identity. The collection  $\Gamma$  is countable and has infinite complement in  $\mathcal{C}(S)$ , the curve complex of  $S$ . As a consequence we obtain that the natural action of the extended mapping class group of  $S$  on  $\mathcal{C}(S)$  is faithful.

## 1 Introduction

Let  $S$  be an orientable surface and  $\text{MCG}^*(S)$  the *extended mapping class group* of  $S$ , that is the group of *all* homeomorphisms of  $S$  modulo isotopy. There is a natural non-trivial action of  $\text{MCG}^*(S)$  on  $\mathcal{C}(S)$ , the curve complex of  $S$ . This is the abstract simplicial complex whose vertices are the (isotopy classes of) essential curves in  $S$ , and whose simplices are multicurves of finite cardinality, see [5] for a detailed discussion. It is well-known that this action is faithful whenever  $S$  has finitely generated fundamental group (except for the closed surface of genus 2), see [7], [9], [8]. The proof of this fact follows from the so-called *Alexander method* (see [5], §2.3 for details). Roughly speaking, this “method” states that the isotopy class of a homeomorphism of  $S$  is often determined by its action on a well-chosen collection of curves and arcs in  $S$ .

The main purpose of this article is to extend the Alexander method for *all* infinite-type surfaces, *i.e.* when  $\pi_1(S)$  is *not* finitely generated. More precisely, we prove:

**Theorem 1.1.** *Let  $S$  be an orientable surface of infinite topological type, with possibly non-empty boundary. There exists a collection of essential arcs and simple closed curves  $\Gamma$  on  $S$  such that any orientation-preserving homeomorphism fixing pointwise the boundary of  $S$  that preserves the isotopy classes of the elements of  $\Gamma$ , is isotopic to the identity.*

As a matter of fact, we give a recipe to construct the collection  $\Gamma$  in §3, from which it is easy to see that  $\Gamma$  is a “small” countable collection of curves (*i.e.*  $\mathcal{C}^0(S) \setminus \Gamma$  is infinite). Moreover, we prove that there are uncountably many different examples of this kind of collections, see lemma 3.8. Recall that any homeomorphism of  $S$  that reverses orientation does not act trivially on  $\mathcal{C}(S)$ , see [10]. As an immediate consequence of Theorem 1.1 we obtain:

**Corollary 1.2.** *Let  $S$  be an orientable surface of infinite topological type with empty boundary. Then the natural action of the extended mapping class group of  $S$  on the curve complex  $\mathcal{C}(S)$  is faithful.*

It is important to remark that the preceding corollary was already known for infinite type surfaces  $S$  for which all ends<sup>1</sup> carry (infinite) genus, see [6]. However, the methods used in that paper to prove that a homeomorphism  $h \in \text{Homeo}(S)$  is isotopic to the identity require that  $h$  fixes *all* isotopy classes in  $\mathcal{C}(S)$ . Therefore the results we present here are both an extension and a refinement of the results obtained in [Ibid.]. Moreover, corollary 1.2 will be used in a second paper<sup>2</sup> to prove that the group of simplicial automorphisms of  $\mathcal{C}(S)$  is naturally isomorphic to  $\text{MCG}^*(S)$  for *any* infinite-type surface. This result is used in [1] to prove that every automorphism of a certain subgroup<sup>3</sup> of  $\text{MCG}^*(S)$  is induced by a homeomorphism of  $S$ , when  $S$  is a genus  $g \geq 0$  closed surface from which a Cantor set has been removed.

**Reader's guide.** In §2 we make a short discussion on infinite-type surfaces and the generalities about curves, arcs and continuous deformations of elements in  $\text{Homeo}^+(S)$ . The real content of the paper is found in §3 where we prove Theorem 1.1. At the end we added an appendix to address some technical aspects of the proofs we present in section 3.

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## 2 Preliminaries

Let  $S$  be an orientable surface. We say  $S$  is a finite-type surface if its fundamental group is finitely generated; otherwise, we say  $S$  is of infinite type.

An infinite-type surface  $S$  is determined, up to homeomorphism, by two invariants: its genus  $g(S) \in \mathbb{N} \cup \{\infty\}$  and a pair of nested topological spaces  $\text{Ends}_\infty(S) \subset \text{Ends}(S)$  forming what is known as *the space of ends* of  $S$ . Roughly speaking, the space  $\text{Ends}(S)$  codifies the different ways in which a surface tends to infinity and  $\text{Ends}_\infty(S)$  is the subspace formed by all those ends that carry (infinite) genus. The spaces  $\text{Ends}(S)_\infty \subset \text{Ends}(S)$  are homeomorphic to a pair of nested closed subsets of the standard triadic Cantor set. Conversely, every infinite-type topological surface can be constructed from such a pair  $X_\infty \subset X$  as follows: think of  $X_\infty \subset X$  as living in the unit interval used to construct the standard triadic Cantor set. Think of the unit interval as a subset of the sphere  $\mathbb{S}^2$  and imagine that this sphere lives in  $\mathbb{R}^3$ . Push points in  $X$  infinitely away from the origin. The resulting surface  $S'$  has a space of ends homeomorphic to  $X$ . Now, for each  $x \in X_\infty \subset X$  add a divergent sequence

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<sup>1</sup>See §2 below for details on the space of ends of a surface.

<sup>2</sup>We decided to write a second paper since the methods used to prove that any simplicial automorphism of  $\mathcal{C}(S)$  is geometric are very different from the ones we present here.

<sup>3</sup>These are called by the authors *asymptotically rigid mapping class groups*.

of handles to  $S'$  that accumulates to  $x$ . This produces a surface  $S$  whose space of ends is homeomorphic to  $X_\infty \subset X$ .

We refer the reader to the work of Ian Richards [12] for a more detailed discussion on the topological classification of infinite-type surfaces. We do not discuss these results further on since we do not need them for the proof of our results.

If  $S$  is a finite-type surface of genus  $g$ ,  $n$  punctures and  $b$  boundary components, we denote by  $\kappa(S) = 3g - 3 + n + b$  the *complexity* of  $S$ . If  $S$  is an infinite-type surface, we define its complexity as infinity.

A *curve* on  $S$  is a topological embedding  $\alpha : \mathbb{S}^1 \rightarrow S$ . We say a curve is *essential* if it is not isotopic to the boundary curve of a neighbourhood of a puncture, to a point, nor to a boundary component. All curves are considered to be essential unless otherwise specified.

An *arc* on  $S$  is a topological embedding  $\alpha : I \rightarrow S$ , where  $I$  is the closed unit interval, and  $\alpha(\partial I) \subset \partial S$ . We consider all isotopies between arcs to be relative to  $\partial I$ , *i.e.* the isotopies are not allowed to move the endpoints. We say an arc is *essential* if it is not isotopic to an arc whose image is completely contained in  $\partial S$ . In this work, all arcs are considered to be essential unless otherwise specified.

We often abuse notation and use the terms “curve” and “arc” to refer to both the topological embeddings and the corresponding images on  $S$ . The context makes clear the meaning in each case.

Let  $[\alpha]$  and  $[\beta]$  be two isotopy classes of essential curves or arcs. The (geometric) *intersection number* of  $\alpha$  and  $\beta$  is defined as follows:

$$i([\alpha], [\beta]) := \min\{ |(\gamma \cap \delta) \cap \text{int}(S)| : \gamma \in [\alpha], \delta \in [\beta] \}.$$

Let  $\alpha$  and  $\beta$  be two essential curves or arcs on  $S$ , we say  $\alpha$  and  $\beta$  are in *minimal position* if  $|(\alpha \cap \beta) \cap \text{int}(S)| = i([\alpha], [\beta])$ .

We denote  $\text{Homeo}^+(S; \partial S)$  the group of orientation-preserving homeomorphisms of  $S$  which fix the boundary pointwise.

We conclude this section with a remark that is fundamental for the rest of the article:

**Remark 2.1** (Equivalence between homotopy and isotopy). *Let  $S$  be an infinite-type surface and  $f, g \in \text{Homeo}(S; \partial S)$ . Then  $f$  is homotopic to  $g$  relative to the boundary if and only if  $f$  is isotopic to  $g$  relative to the boundary.*

To simplify the exposition, henceforth we suppose that all homotopies and isotopies in this text are relative to the boundary.

For surfaces of finite-topological type the result is well-known, see [2], [4] and Theorem 1.12 in [5]. For infinite-type surfaces, the results follow from the work of Cantwell and Conlon, see [3].

### 3 Alexander method

We begin this section by introducing the notion of an *Alexander system* and then some technical lemmas. We then introduce the notion of a *stable* Alexander system. These are

collections of arcs and curves such that any  $f \in \text{Homeo}^+(S; \partial S)$  that preserves the isotopy classes of their elements is isotopic to the identity. In the last part of this section we prove theorem 1.1, which states that stable Alexander systems exist for infinite type surfaces. We finish the section by showing that for any infinite-type surface  $S$  there are actually *uncountably* many different stable Alexander systems in  $S$ .

**Definition 3.1.** Let  $\Gamma = \{\gamma_i\}_{i \in I}$  be a collection of essential curves and arcs on  $S$ . We say  $\Gamma$  is an Alexander system if it satisfies the following conditions:

1. The elements in  $\Gamma$  are in pairwise minimal position.
2. For  $\gamma_i, \gamma_j \in \Gamma$  with  $i \neq j$ , we have that  $\gamma_i$  is not isotopic to  $\gamma_j$ .
3. For all distinct  $i, j, k \in I$ , at least one of the following sets is empty:  $\gamma_i \cap \gamma_j$ ,  $\gamma_j \cap \gamma_k$ ,  $\gamma_k \cap \gamma_i$ .

Note that any subset of an Alexander system is also an Alexander system and that we do not require the surface  $S$  in the preceding definition to be of infinite type. The following result is the infinite-type surface version of Proposition 2.8 in [5], to the point that its proof is also completely analogous.

**Lemma 3.2.** Let  $S$  be a connected orientable surface of infinite topological type with possibly nonempty boundary, and  $\Gamma = \{\gamma_1, \dots, \gamma_k\}$  be a finite Alexander system on  $S$ . If  $h \in \text{Homeo}^+(S, \partial S)$  is such that for all  $i = 1, \dots, k$  we have that  $h(\gamma_i)$  is isotopic to  $\gamma_i$ , then there exists  $f \in \text{Homeo}^+(S, \partial S)$  isotopic to the identity on  $S$  relative to  $\partial S$ , such that  $f|_\Gamma = h|_\Gamma$ .

**Remark 3.3.** Note that if  $S$  is not connected and  $\Gamma$  has only finitely many elements on each connected component of  $S$ , we can apply either Proposition 2.8 in [5] or Lemma 3.2 above to each connected component.

Both Lemma 3.2 and Remark 3.3 are used repeatedly in the proofs that we present.

One of the ideas in these proofs is to use convenient families of subsurfaces that exhaust a fixed infinite-type surface. We introduce these below.

**Definition 3.4.** Let  $\{S_i\}$  be an (set-theoretical) increasing sequence of subsurfaces of  $S$ . We say  $\{S_i\}$  is a principal exhaustion of  $S$  if  $S = \bigcup_{i \geq 1} S_i$  and for all  $i \geq 1$  it satisfies the following conditions:

1.  $S_i$  is a surface of finite topological type,
2.  $S_i$  is contained in the interior of  $S_{i+1}$ ,
3.  $\partial S_i - \partial S$  is the finite union of pairwise disjoint essential curves on  $S$ , and
4. each connected component of  $S_{i+1} \setminus S_i$  has complexity at least 6.

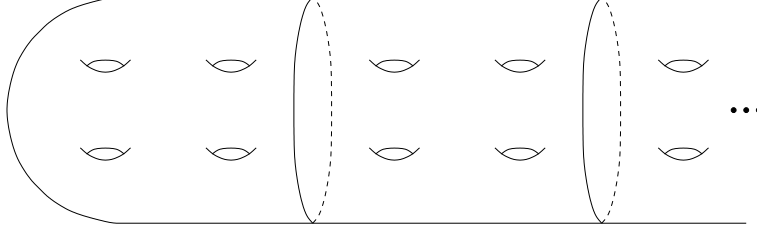


Figure 1: An example of a principal exhaustion for the “Loch Ness Monster”, i.e. the surface of infinite genus and one end.

Note that these exhaustions always exist in any infinite-type surface. See Figures 1 and 2 for examples.

Let  $\{S_i\}$  be a principal exhaustion of  $S$ . For each  $i \geq 1$ , we denote by  $B_i$  the set of the boundary curves of  $S_i$  that are essential curves on  $S$ . Note that for  $i \neq j$  we have that  $B_i \cap B_j = \emptyset$ . We define the *boundaries of  $\{S_i\}$* , denoted by  $B$ , as  $B := \bigcup_{i \geq 1} B_i$ .

The first step of the proof of Theorem 1.1 is the following.

**Lemma 3.5.** *Let  $S$  be an orientable surface of infinite topological type,  $\{S_i\}$  a principal exhaustion of  $S$ , and  $B$  be the boundaries of  $\{S_i\}$ . If  $h \in \text{Homeo}^+(S; \partial S)$  is such that  $h(\gamma)$  is isotopic to  $\gamma$  for every  $\gamma \in B$ , then  $h$  is isotopic to a homeomorphism  $g \in \text{Homeo}^+(S; \partial S)$  for which  $g|_B = \text{id}_S|_B$ .*

*Proof.* We divide the proof into two parts. In the first part we construct the homeomorphism  $g$  using Lemma 3.2 and Remark 3.3. As a consequence of the construction we have that  $g|_B = \text{id}_S|_B$ . In the second part, using classical results from obstruction theory (see the appendix and references therein), we show that  $h$  and  $g$  are isotopic.

**First part: the construction of  $g$ .** Remark that for each integer  $i \geq 1$ ,  $B_i$  is a finite Alexander system, both on  $S$  and on  $S \setminus S_k$  for each  $k < i$ . Since both  $h$  and  $B_1$  satisfy the conditions of Lemma 3.2, there exists  $f_1 \in \text{Homeo}^+(S; \partial S)$  isotopic to  $\text{id}_S$  such that  $f_1|_{B_1} = h|_{B_1}$ . Let  $g_1 := f_1^{-1} \circ h$ . Then  $g_1|_{B_1} = \text{id}_S|_{B_1}$ . As a consequence, we have that  $g_1(S_1) = S_1$ , and  $g_1$  also fixes the isotopy classes of  $B$ . This implies that both the restriction to  $S \setminus \text{int}(S_1)$  of  $g_1$ , and  $B_2$  satisfy the conditions of Remark 3.3 on  $S \setminus \text{int}(S_1)$ .

Let  $\tilde{g}_1 := g_1|_{S \setminus \text{int}(S_1)}$ , then by Remark 3.3, there exists  $\tilde{f}_2 \in \text{Homeo}^+(S \setminus \text{int}(S_1); \partial S)$  isotopic to  $\text{id}_{S \setminus \text{int}(S_1)}$  relative to  $\partial(S \setminus \text{int}(S_1))$  such that  $\tilde{f}_2|_{B_2} = \tilde{g}_1|_{B_2}$ . Thus, we define the following:

$$f_2(s) = \begin{cases} s & \text{if } s \in S_1, \\ \tilde{f}_2(s) & \text{otherwise.} \end{cases}$$

Note that  $f_2$  is also isotopic to  $\text{id}_S$ . Then, we define  $g_2 := f_2^{-1} \circ g_1 \in \text{Homeo}^+(S; \partial S)$ .

By construction,  $g_2$  satisfies that  $g_2|_{B_2} = \text{id}_S|_{B_2}$ , and  $g_2|_{S_1} = g_1|_{S_1}$ . Moreover,  $g_2$  preserves the connected components of  $S \setminus S_2$ , and  $g_2$  is isotopic to  $g_1$ .

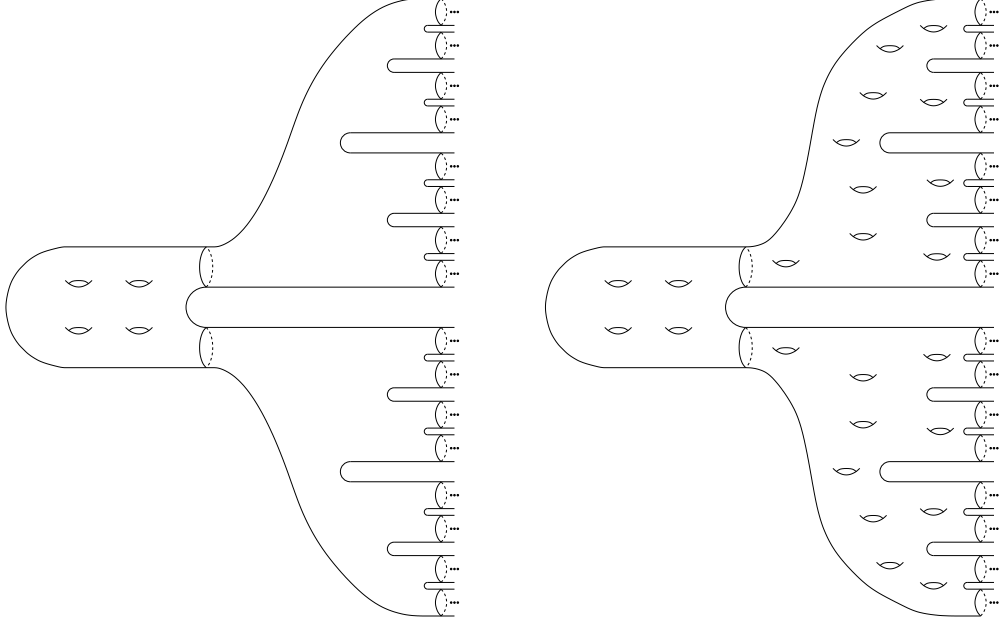


Figure 2: On the left, an example of a principal exhaustion for the surface of genus 4 minus a Cantor set. On the right, an example of a principal exhaustion for a surface of infinite genus, and whose space of ends that carry genus is a Cantor set.

In this manner, we inductively define for each  $n > 2$  a homeomorphism  $g_n \in \text{Homeo}^+(S; \partial S)$  such that:

1.  $g_n$  is isotopic to  $g_{n-1}$ ,
2.  $g_n|_{B_n} = \text{id}_S|_{B_n}$ ,
3. for all  $m < n$ , we have that  $g_n|_{S_m} = g_m|_{S_m}$ .

We define the following map:

$$\begin{aligned} g : S &\rightarrow S \\ s &\mapsto g_n(s) \text{ if } s \in S_n. \end{aligned}$$

This map is well-defined since all  $g_n$  satisfy (3) above. Also, by construction we have that  $g \in \text{Homeo}^+(S; \partial S)$  and  $g|_B = \text{id}_S|_B$ .

**Second part: the map  $g$  is isotopic to  $h$ .** We claim that for every  $n \geq 1$  the map  $g_n$  is homotopic to  $g_{n+1}$  relative to  $S_n$ , that is, there exist a homotopy  $H_n : S \times I \rightarrow S$  satisfying:

- (a)  $H_n|_{S \times \{0\}} = g_n$ ,
- (b)  $H_n|_{S \times \{1\}} = g_{n+1}$  and
- (c)  $H_n(x, t) = g_n(x) = g_{n+1}(x)$  for every  $t \in I$  and  $x \in S_n$ .

We postpone the proof of this claim to the appendix.

For each  $n \geq 1$ , we define the homeomorphism

$$\begin{aligned} \zeta_n: \quad \left[ \frac{n-1}{n}, \frac{n}{n+1} \right] &\rightarrow [0, 1] \\ t &\mapsto n(n+1) \left( t - \frac{n-1}{n} \right). \end{aligned}$$

By conditions (a) and (b) over the family of homotopies  $\{H_n\}$  the map  $H : S \times [0, 1] \rightarrow S$  given by

$$H(s, t) = \begin{cases} H_n(s, \zeta_n(t)) & \text{if } t \in \left[ \frac{n-1}{n}, \frac{n}{n+1} \right], \\ g(s) & \text{if } t = 1. \end{cases}$$

is well-defined. We claim that  $H$  is a homotopy between  $g_1$  and  $g$ . First, we note that for all  $s \in S$ ,  $H(s, 0) = H_1(s, \zeta_1(0)) = H_1(s, 0) = g_1(s)$  and  $H(s, 1) = g(s)$ . It remains to prove that  $H$  is continuous. If  $(s, t) \in S \times [0, 1)$ , then by definition, for some  $n \geq 1$ ,  $H$  and  $H_n$  coincide in some open neighborhood of  $(s, t)$ . By the continuity of  $H_n$ ,  $H$  is continuous at  $(s, t)$ . Now, let  $(s, 1) \in S \times \{1\}$ . Then there exists  $n \geq 1$  such that  $s \in \text{int}(S_m)$  for all  $m \geq n$ . Choose  $U_s$  an open neighborhood of  $s$  properly contained in  $\text{int}(S_m)$ . By condition (c), for all  $m \geq n$  and  $(s', t') \in U_s \times \left( \frac{n-1}{n}, 1 \right]$ , we have that  $H(s', t') = H_m(s', t') = g_m(s') = g(s')$ . Thus,  $H$  coincides with  $g$  in some open neighborhood of  $(s, 1)$  and we deduce that  $H$  is continuous at  $(s, 1)$ .

Finally, since  $h$  is isotopic to  $g_1$ , we obtain that  $h$  is homotopic to  $g$ . From Remark 2.1 we conclude that  $h$  is isotopic to  $g$ , as desired.  $\square$

Let  $N$  be a subsurface of  $S$ , and  $A$  be a collection of curves and arcs on  $S$  whose image is contained in  $N$ . We say  $A$  *fills*  $N$  if  $\text{int}(N \setminus A)$  is the disjoint union of open discs and once-punctured open discs.

**Definition 3.6.** *Let  $\Gamma$  be an Alexander system on  $S$ . We say  $\Gamma$  is a stable Alexander system if  $\Gamma$  fills  $S$  and every  $f \in \text{Homeo}^+(S; \partial S)$  that preserves the isotopy classes of elements in  $\Gamma$  is isotopic relative to the boundary, to the identity.*

**Remark 3.7.** *In the preceding definition we do not require  $S$  to be of infinite type. Also note that not every Alexander system that fills  $S$  is stable. See Figures 3 and 4 for a counterexamples and an example of stable Alexander systems on finite-type surfaces respectively.*

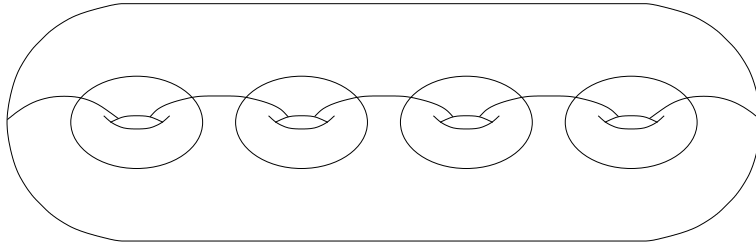


Figure 3: An Alexander system that fills  $S$  but is not stable.

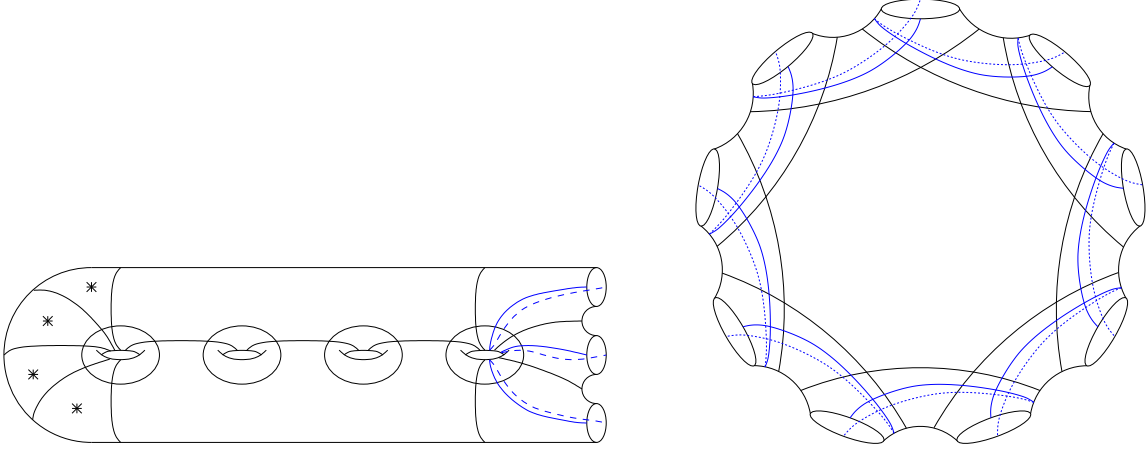


Figure 4: On the left, an example of a stable Alexander system for a surface of genus 4, with 4 punctures and 3 boundary components. On the right, an example of a stable Alexander system for a surface of genus 0 with 9 boundary components.

**Proof of Theorem 1.1.** First we construct an Alexander system  $\Gamma$  that fills  $S$  and then we show that this system is stable.

Let  $\{S_n\}$  be a principal exhaustion of  $S$ , and  $B$  be the boundaries of  $\{S_n\}$ . For each  $n \geq 1$ , choose a finite stable Alexander system  $C_n$  for  $\text{int}(S_n \setminus S_{n-1})$  (with  $S_0 = \emptyset$ ) and let  $C$  be the union of the collection of  $\{C_n\}$ . See figure 4 for examples of stable Alexander systems.

Note that  $B \cup C$  is not a stable Alexander system. There are homeomorphisms which are not homotopic to the identity but that fix all the (isotopy classes of) elements in this collection, *e.g.* any representative of a Dehn-twist along a curve in  $B$ . As we prove below, these are the only homeomorphisms showing this behaviour.

Let  $B^*$  be a collection of curves, and  $f : B \rightarrow B^*$  a bijection satisfying that for all  $\gamma \in B$  and all  $\delta \in B \setminus \{\gamma\}$ ,  $i([\gamma], [f(\gamma)]) \neq 0$  and  $i([\delta], [f(\gamma)]) = 0$ . Remark that we can choose  $B^*$  so that  $\Gamma := B \cup B^* \cup C$  is an Alexander system. Figure 5 illustrates how to choose the collection  $B^*$ . We finish this proof by showing that  $\Gamma$  is a *stable* Alexander system.

Let  $h \in \text{Homeo}^+(S; \partial S)$  be such that  $h$  fixes the isotopy class of every curve in  $\Gamma$ . We claim that  $h$  is isotopic to the identity on  $S$ .

First, since  $h$  fixes the isotopy class of elements in  $B \subset \Gamma$ , by the Lemma 3.5,  $h$  is isotopic to some  $g \in \text{Homeo}^+(S; \partial S)$  such that  $g|_B = \text{id}_S|_B$ . In particular,  $g$  also fixes all the isotopy classes of elements in  $\Gamma$ .

Observe that for  $n \geq 1$ ,  $S_n \setminus \text{int}(S_{n-1})$  is a disjoint union of a finite number of finite-type surfaces with boundary, and by construction  $(S_n \setminus \text{int}(S_{n-1})) \cap \Gamma$  is a finite stable Alexander system on  $S_n \setminus \text{int}(S_{n-1})$ . Then by definition of stable Alexander systems, the restriction of  $g$  to  $S_n \setminus \text{int}(S_{n-1})$  is isotopic to the identity on  $S_n \setminus \text{int}(S_{n-1})$  relative to the boundary  $\partial(S_n \setminus \text{int}(S_{n-1}))$ . Given that  $g|_B = \text{id}_S|_B$ , the homeomorphism  $g$  is isotopic to the identity on  $S$ .  $\square$



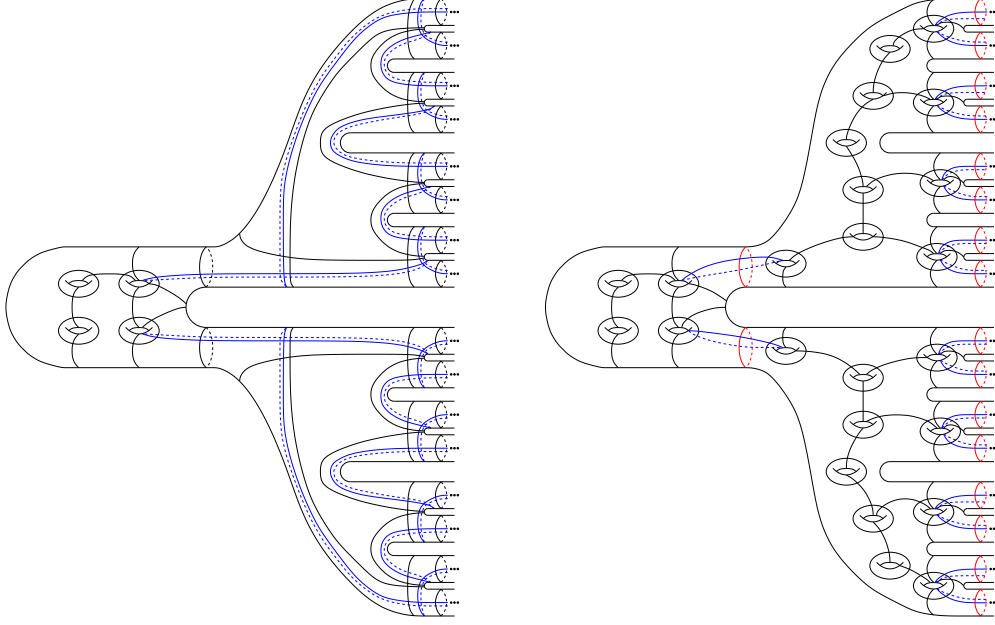


Figure 5: On the left, an example of  $\Gamma$  for the surface of genus 4 minus a Cantor set. On the right, an example of  $\Gamma$  for a surface of infinite genus, and whose space of ends that carry genus is a Cantor set.

**Lemma 3.8.** *Let  $S$  be an orientable connected surface of infinite topological. Then there exists uncountably many isotopy classes of stable Alexander systems on  $S$ .*

*Proof.* Fix  $\{S_i\}$  a principal exhaustion of  $S$ , and let  $\Gamma = B \cup C \cup B^*$  be as in the proof of Theorem 1.1. For each  $i \geq 1$  let  $\varphi_i \in \text{Homeo}^+(\text{int}(S_i \setminus S_{i-1}); \partial S)$  (with  $S_0 = \emptyset$ ) be a map that preserves each connected component of  $\text{int}(S_i \setminus S_{i-1})$  and whose restriction to each connected component is a pseudo-Anosov map.

Let  $\mathcal{A}$  denote the set of all isotopy classes of stable Alexander systems of  $S$ . Note that  $\mathcal{A}$  is invariant under the action of  $\text{Homeo}^+(S; \partial S)$ ; hence we can define the following map:

$$\begin{aligned} \Phi : \{0, 1\}^{\mathbb{Z}^+} &\rightarrow \mathcal{A} \\ (\epsilon_i)_{i \geq 1} &\mapsto \left[ B \cup \left( \bigcup_{i \geq 1} \varphi_i^{\epsilon_i}(C_i) \right) \cup \left( \bigcup_{i \geq 1} \varphi_i^{\epsilon_i}(f(B_i)) \right) \right]; \end{aligned}$$

where  $f$  is the bijection between  $B$  and  $B^*$ . Since each  $\varphi_i$  is a pseudo-Anosov map, the map  $\Phi$  is injective. Thus,  $\mathcal{A}$  is an infinitely uncountable set.  $\square$

## A Appendix.

In this appendix we explain how classical results of obstruction theory are used to assure the existence of the homotopies  $H_n$  used in the proof of lemma 3.5. Our discussion is based on

the work of P. Olum [11] and adapted to the context of orientable surfaces with boundary.

*The extension problem.* Let  $S$  be an orientable surface<sup>4</sup> and  $S' \subset S$  a subsurface. Consider two continuous functions  $f_0, f_1 : S \rightarrow S$  such that  $f_0(s) = f_1(s)$  for all  $s \in S'$ . Following [11], let  $\overline{S}'_{01}$  be the subset of  $S \times I$  formed by  $S \times 0 \cup S' \times I \cup S \times \{1\}$ . Define  $F : \overline{S}'_{01} \rightarrow S$  by:

$$F(s, t) = \begin{cases} f_0(s) & \text{for } (s, t) \in S \times \{0\} \cup S' \times I \\ f_1(s) & \text{for } (s, t) \in S \times \{1\} \end{cases} \quad (1)$$

We say that  $f_0$  is *homotopic to  $f_1$  relative to  $S'$*  if  $F$  defined above has a continuous extension to  $S \times I$ . The following result is an adaptation of theorem 25.2 in [Ibid.] to the context of orientable surfaces with boundary; it gives criteria to determine when  $f_0$  and  $f_1$  are homotopic relative to  $S'$ . We state the result first, then explain its content and finally show how to apply to the proof of lemma 3.5.

**Theorem A.1.** *Let  $f_0, f_1 : S \rightarrow S$  be two continuous functions such that  $f_0(s) = f_1(s)$  for all  $s \in S'$ . Let  $s_0 \in S'$  and  $\theta_i : \pi_1(S, s_0) \rightarrow \pi_1(S, f_i(s_0))$  be the homomorphism induced by  $f_i$ , for  $i = 1, 2$ . For  $k \in \{0, 1, 2\}$  fixed the following statements are equivalent:*

1. (For  $k = 2$ )  $\mathbf{O}^k(f_0, f_1) \text{ rel. } S'$  is non-void and contains the zero element.
2. (For  $1 \leq k$ )  $f_0 \simeq f_1 \text{ dim } k \text{ (rel } S')$ .
3. (For  $1 \leq k$ )  $\mathbf{O}^{k+1}(f_0, f_1) \text{ rel. } S'$  is non-void.
4. The homomorphism  $\theta_0$  and  $\theta_1$  induced by  $f_0$  and  $f_1$  are equal.

We now explain the content of this theorem. Let  $\tau$  be a triangulation of  $S$  and  $\tau^k$  its  $k$ -skeleton (in particular  $\tau^2 = S$ ). Two functions  $f_0$  and  $f_1$  satisfying the hypothesis of the preceding theorem are *homotopic in dimension  $k$  relative to  $S'$* , written  $f_0 \simeq f_1 \text{ dim } k \text{ (rel } S')$ , if  $f_0$  restricted to  $S' \cup \tau^k$  is homotopic to  $f_1$  restricted to  $S' \cup \tau^k$  relative to  $S'$ . This is precisely the content of point (2) above. In points (1) and (3) appears  $\mathbf{O}^k(f_0, f_1) \text{ rel. } S'$ , the  $k^{\text{th}}$  obstruction to a homotopy of  $f_0$  to  $f_1$  relative to  $S'$ . This is a subset of the cohomology group  $H^k(S, S', \theta_0^* \pi_n)$ . Here  $\theta_0^* \pi_k$  is the system of (twisted) local groups  $\theta_0^* \pi_k(S, s_0)$ , where  $\pi_k(S, s_0)$  is the  $k^{\text{th}}$ -homotopy group of  $S$  with based at  $s_0$ . The definition of both the obstruction and the cohomology groups is well explained in [Ibid.], but as we will see we do not need them because in our context all these objects are trivial.

*Constructing the homotopies  $H_n$ .* Recall from the proof of lemma 3.5 that for each  $n \in \mathbb{N}$  we have two isotopic maps  $g_n, g_{n+1} \in \text{Homeo}^+(S; \partial S)$  which coincide in  $S_n \subset S$ . Since the maps are isotopic, the isomorphisms that they induce in a fundamental group of  $S$  with basepoint in  $S_n$  are equal. In other words, these maps satisfy (4) in theorem A.1.

By (3) above we have that  $\mathbf{O}^2(f_0, f_1) \text{ rel. } S'$  is non-void. Since  $S$  is not a sphere, the cohomology groups  $H^2(S, S_n, \theta_0^* \pi_n)$  are trivial because all local coefficients  $\pi_2(S, s_0)$  are trivial. Therefore,  $\mathbf{O}^2(f_0, f_1)$  only contains the zero element, satisfying thus (1) for  $k = 2$ . In particular, by (2) above we can conclude that  $g_n$  and  $g_{n+1}$  are homotopic in dimension 2 relative to  $S'$ , which is precisely what we wanted.

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<sup>4</sup> $S$  is not necessarily compact, nor necessarily of finite-type and its boundary might be non-empty.

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